

Math 2250 Lab 9

Due Date: 3/26/2015

Name: _____

1. Floating Bodies and the Harmonic Oscillator.

Archimedes' principle states that a body partially or totally submerged in a liquid is buoyed up by a force equal to the weight of the liquid displaced. The fact that the body is resting or floating on water implies that the downward force due to the weight $W = mg$ of the body must be the same as the upward buoyant force of the water, namely the weight of the water displaced. Call $y = 0$ the equilibrium position of the lower end of a body when it is floating in a liquid. If the body is now depressed a distance y from its equilibrium position, an additional upward force will act on the body due to the additional liquid displaced. If the body's cross-sectional area is A , then the volume of additional liquid displaced is $(Ay)\rho$, where ρ is the weight per unit volume of liquid. Because this additional upward force is the net force acting on the depressed body, then Newton's second law of motion *force = mass \times acceleration* implies

$$m \frac{d^2 y}{dt^2} = -\rho A y,$$

where the positive y -direction is **downward**. This equation is the classical **harmonic oscillator** $y''(t) + \omega^2 y(t) = 0$ with $\omega = \sqrt{\rho A / m}$. A floating body therefore undergoes simple harmonic motion when depressed from its equilibrium position.

Problem 1. Suppose a body of mass m has a cross-sectional area A sq ft. The body is pushed downward y_0 ft from its equilibrium position in a liquid whose weight per ft³ is ρ and released.

(a) Show details for the following result: The model is

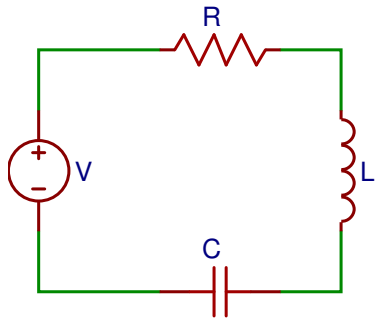
$$\begin{cases} m \frac{d^2 y}{dt^2} + \rho A y = 0, \\ y(0) = y_0, \quad y'(0) = 0, \end{cases} \quad (1)$$

with unique solution $y(t) = y_0 \cos \left(\sqrt{\frac{\rho A}{m}} t \right)$.

- (b) Assume the body is a right circular cylinder with vertical axis along the y -axis and radius r . Show that the period is $T = \frac{2}{r} \sqrt{\frac{m\pi}{\rho}}$.

References: Edwards-Penney Section 5.1

2. The LC-Circuit Equation and the Method of Undetermined Coefficients.



Consider the series RLC circuit in the figure, in which the resistor R has been removed ($R = 0$ assumed).

Assumed are zero initial charge $Q(0) = 0$ and zero initial current $Q'(0) = 0$.

Suppose a periodic voltage source $V_0 \cos(\omega t)$ is applied to the circuit. Using Kirchhoff's laws, the following initial value problem gives the charge $Q(t)$ on the capacitor.

$$\begin{cases} L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = V_0 \cos(\omega t), \\ Q(0) = 0, \quad Q'(0) = 0. \end{cases}$$

Take $L = 1 \text{ V} \cdot \text{s} \cdot \text{A}^{-1}$, $R = 0 \text{ } \Omega$, $C^{-1} = 0.25 \text{ V} \cdot \text{C}^{-1}$, and $V_0 = 4 \text{ V}$. The left sides of these equations define symbols L , R , C , V_0 using physical units V =volts, s =seconds, A =amperes, Ω =ohms, C =coulombs.

Angular frequency is the coefficient ω_0 in trigonometry terms such as $\cos(\omega_0 t)$, $\cos(\omega_0 t - \alpha)$, $\sin(\omega_0 t)$, $\sin(\omega_0 t - \alpha)$. Angular frequency (natural frequency) may differ from references to *frequency* used in physics and engineering courses.

- (a) Find the angular frequency ω_0 of this system, by determining the angular frequency ω_0 of solutions to the unforced equation

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = 0.$$

- (b) Assume that $\omega \neq \omega_0$. Let symbol $Q_H(t)$ be the solution to the homogeneous equation, containing arbitrary constants c_1, c_2 . Use the *method of undetermined coefficients* to solve for a particular solution $Q_P(t)$. Then use $Q(t) = Q_P(t) + Q_H(t)$ to solve the initial value problem for $Q(t)$ (evaluate c_1, c_2 using $Q(0) = Q'(0) = 0$). Check the answer $Q(t)$ with technology.
- (c) Write down the solution $Q(t)$ found in part b) for $\omega = 0.6$, which is a superposition of two cosine functions. Compute the period of this solution. Use technology to graph $Q(t)$ for one period.
- (d) Now let $\omega = \omega_0$. Use the *method of undetermined coefficients* to solve for a new particular solution $Q_P(t)$. Then use $Q(t) = Q_P(t) + Q_H(t)$ to solve the initial value problem for $Q(t)$.

- (e) The phenomenon of **Beats** and the phenomenon of **Pure Resonance** appear in the solutions obtained above. Explain fully (This is not included in the worksheet on Thursday, but it will be beneficial to think about it. You should also refer to Edwards-Penney Section 5.6).

References: Edwards-Penney Section 5.1, 5.5, 5.6.